Preconstructibility of tempered solutions of holonomic $\mathcal{D}$-modules

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Abstract

In this paper we prove the preconstructibility of the complex of tempered holomorphic solutions of holonomic $\mathcal{D}$-modules on complex analytic manifolds. This implies the finiteness of such complex on any relatively compact open subanalytic subset of a complex analytic manifold. Such a result is an essential step for proving a conjecture of M. Kashiwara and P. Schapira ([13]) on the constructibility of such complex.

Contents

Introduction **2**

1 Notations and review **4**
- 1.1 Subanalytic sites . . . 4
- 1.2 $\mathcal{D}$-modules and tempered holomorphic functions . . . 6
- 1.3 Elementary asymptotic decomposition of meromorphic connections . . . 10

2 $\mathbb{R}$-preconstructibility of the tempered De Rham complex for holonomic $\mathcal{D}$-modules **13**
- 2.1 The case of modules supported on analytic sets . . . 14
- 2.2 The case of meromorphic connections . . . 15
- 2.3 The general statement . . . 21

References **22**

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Introduction

The Riemann–Hilbert correspondence is one of the more powerful results in $\mathcal{D}$-module theory. It has important applications in many fields of mathematics such as representation theory or microlocal analysis. Such a correspondence gives an equivalence between the category of *regular* holonomic $\mathcal{D}$-modules on an analytic manifold X and the category of perverse sheaves on X. The former is of analytic nature, as it is a categorical description of certain linear partial differential systems. The latter is of topological nature as it is defined using subanalytic stratifications of X and some combinatorial properties. Such an equivalence is realized through the complex of holomorphic solutions of regular holonomic $\mathcal{D}$-modules. In the proof of the Riemann–Hilbert correspondence given by M. Kashiwara ([8, 9]) and, more generically, in the study of the complex of holomorphic solutions of a holonomic $\mathcal{D}$-module, the property of constructibility of such complex is fundamental ([7]). Let us simply recall that a bounded complex of sheaves is constructible if its cohomology groups are locally constant sheaves on the strata of an analytic stratification and if their stalks have finite dimension.

The study of *irregular* holonomic $\mathcal{D}$-modules is much more complicated. In dimension 1, a local irregular Riemann–Hilbert correspondence was proved through the works of many mathematicians as P. Deligne, M. Hukuhara, A. Levelt, B. Malgrange, J.-P. Ramis, Y. Sibuya, H. Turrittin (see [3] and [18]). Roughly speaking, such a correspondence is obtained in two steps. First one studies the formal structure of flat meromorphic connections (Levelt–Turrittin Theorem) and then one analyses the gluing of the sectorial asymptotic lifts of the formal structures. The higher dimensional case is still open. C. Sabbah conjectured the analogue of the Levelt–Turrittin Theorem and he proved it in some particular cases ([25]). Furthermore, he deeply studied the asymptotic lifting property ([24]). In [19] and [20], T. Mochizuki proved Sabbah's conjecture in the algebraic cases. Recently ([14], [15]), K. Kedlaya proved Sabbah's conjecture in the analytic case. In dimension 1, the formal decomposition is given by the Levelt–Turrittin Theorem. In higher dimension it is not straightforward and one needs a finer study of the formal invariants of a flat meromorphic connection. This leads to introduce the notion of *good* model which plays a central role in the cited works. These new results should allow a formulation of the irregular Riemann–Hilbert correspondence in higher dimension. Even if the local 1-dimensional case is nowadays classical, it has some points which do not allow the passage to a global description of holonomic $\mathcal{D}$-modules or to a natural generalization to the higher dimensional case. In particular, the notion of the formal invariants of a flat meromorphic connection in dimension 1 has a purely analytic nature and the higher dimensional analogues of such invariants need to satisfy a *goodness* property.

Recently ([13]), M. Kashiwara and P. Schapira introduced the subanalytic site relative to X, denoted X_{sa}, and the complex of sheaves on it of tempered holomorphic functions, denoted $\mathcal{O}^t_{X_{sa}}$. The study of solutions of holonomic $\mathcal{D}$-modules on complex curves with values in $\mathcal{O}^t_{X_{sa}}$ allowed to describe faithfully and in a topological way the formal invariants of flat meromorphic connections ([22]). Furthermore, Kashiwara–Schapira defined a notion of $\mathbb{R}$-constructibility for sheaves on X_{sa} and they conjectured the $\mathbb{R}$-constructibility of tempered solutions of holonomic $\mathcal{D}$-modules. In [21], we proved the conjecture for holonomic $\mathcal{D}$-modules on complex curves. In the present article we prove the $\mathbb{R}$-preconstructibility of the complex of tempered solutions of holonomic $\mathcal{D}$-modules on analytic manifolds. This implies the finiteness of such complex on any relatively compact open subanalytic subset of a complex analytic manifold.

The article is organized as follows. In the first section we review classical results on sheaves on subanalytic sites, $\mathcal{D}$-modules, tempered solutions and elementary asymptotic decompositions of flat meromorphic connections. In this paper, the notion of good model and good decomposition is not needed. Hence, we will simply recall the elementary asymptotic decomposition of flat meromorphic connections without any goodness property. In the second section we state and prove our main result of $\mathbb{R}$-preconstructibility of the tempered De Rham complex of holonomic $\mathcal{D}$-modules. The proof is divided in two parts. In the first part we prove our result for modules supported on a closed analytic subset Z of X by using induction on the dimension of Z and reducing to the 1-dimensional case. In the second part, we consider flat meromorphic connections. We start by proving the $\mathbb{R}$-preconstructibility of the tempered De Rham complex of an elementary model using the results in dimension 1 and some properties on the pull-back of the tempered De Rham complex. Then, we prove the case of a generic flat meromorphic connections by using the results of Kedlaya, Mochizuki and Sabbah.

Let us conclude the introduction recalling that many spaces of functions with growth conditions have been used in the study of irregular $\mathcal{D}$-modules. For example, in [18], the sheaves of holomorphic functions with asymptotic expansion or moderate and Gevrey growth at the origin have been studied. In higher dimension, sheaves of functions with moderate growth or with asymptotic expansion on a divisor Z of X have been used in [24], [25], [5] and [4]. Such sheaves are defined on the real blow-up of X along Z and they do not allow to treat globally holonomic $\mathcal{D}_X$-modules. The $\mathbb{R}$-preconstructibility of the tempered De Rham complex of an elementary model, obtained in the present article (see Lemma 2.2.5), is related to the results, obtained by M. Hien in [5] and [4], on the vanishing of the moderate De Rham complex of a good model. Our approach is quite different to Hien's one. Indeed, Hien proved his results of finiteness and vanishing of the moderate De Rham complex of good models by estimating the growth of some integrals appearing

in the solutions on multisectors (see [24] for the asymptotic expansion case). In our case the situation is more complicated. Indeed, with Hien's approach we should have proved growth estimates of solutions on arbitrary subanalytic open sets. We used this method in dimension 1 ([21]) but in higher dimension the geometry of a subanalytic set can be much more complicated. Hence we adopted a different approach to prove the $\mathbb{R}$-preconstructibility of the tempered De Rham complex of an elementary model. We used our results in dimension 1, combining it with some formulas on the pull-back of the tempered De Rham complex proved in [12]. Remark that, adapting the techniques of [6], one can obtain holomorphic functions with moderate growth from tempered holomorphic functions in a functorial way.

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1 Notations and review

1.1 Subanalytic sites

Let X be a real analytic manifold countable at infinity.

Definition 1.1.1. *(i) A set $Z \subset X$ is said* semi-analytic at $x \in X$ *if the following condition is satisfied. There exists an open neighborhood W of x such that $Z \cap W = \cup_{i\in I} \cap_{j\in J} Z_{ij}$ where I and J are finite sets and either $Z_{ij} = \{y \in X;\ f_{ij}(y) > 0\}$ or $Z_{ij} = \{y \in X;\ f_{ij}(y) = 0\}$ for some real-valued real analytic functions f_{ij} on W. Furthermore, Z is said* semi-analytic *if Z is semi-analytic at any $x \in X$.*

(ii) A set $Z \subset X$ is said subanalytic *if the following condition is satisfied. For any $x \in X$, there exist an open neighborhood W of x, a real analytic manifold Y and a relatively compact semi-analytic set $A \subset X \times Y$ such that $\pi(A) = Z \cap W$, where $\pi : X \times Y \to X$ is the projection.*

Given $Z \subset X$, denote by $\mathring{Z}$ (resp. $\overline{Z}$, ∂Z) the interior (resp. the closure, the boundary) of Z.

Proposition 1.1.2 (See [1]). *Let Z and V be subanalytic subset of X. Then $Z \cup V$, $Z \cap V$, $\overline{Z}$, $\mathring{Z}$ and $Z \setminus V$ are subanalytic. Moreover the connected components of Z are subanalytic, the family of connected components of Z is locally finite and Z is locally connected at any point in Z.*

Definition 1.1.3. *A family $\{A_\alpha\}_{\alpha\in\Lambda}$ of subanalytic subsets of X is said a* stratification of X *if $\{A_\alpha\}_{\alpha\in\Lambda}$ is locally finite, $X = \bigsqcup_{\alpha\in\Lambda} A_\alpha$ and each A_α is a locally closed subanalytic manifold.*

For the theory of sheaves on topological spaces we refer to [11]. For the theory of sheaves on the subanalytic site that we are going to recall now, we refer to [12], see also [23].

We denote by $\mathrm{Op}(X)$ the family of open subsets of X. For k a commutative ring we denote by k_X the constant sheaf. For a sheaf in rings $\mathcal{R}_X$, we denote by $\mathrm{Mod}(\mathcal{R}_X)$ the category of sheaves of $\mathcal{R}_X$-modules on X and by $D^b(\mathcal{R}_X)$ the bounded derived category of $\mathrm{Mod}(\mathcal{R}_X)$.

Let us recall the definition of the subanalytic site X_{sa} associated to X. An element $U \in \mathrm{Op}(X)$ is an open set for X_{sa} if it is open, relatively compact and subanalytic. The family of open sets of X_{sa} is denoted $\mathrm{Op}^c(X_{sa})$. For $U \in \mathrm{Op}^c(X_{sa})$, a subset S of the family of open subsets of U is said an open covering of U in X_{sa} if $S \subset \mathrm{Op}^c(X_{sa})$ and, for any compact K of X, there exists a finite subset $S_0 \subset S$ such that $K \cap (\cup_{V \in S_0} V) = K \cap U$. The set of coverings of U in X_{sa} is denoted by $\mathrm{Cov}_{sa}(U)$.

We denote by $\mathrm{Mod}(k_{X_{sa}})$ the category of sheaves of k-modules on the subanalytic site associated to X. With the aim of defining the category $\mathrm{Mod}(k_{X_{sa}})$, the adjective "relatively compact" can be omitted in the definition above. Indeed, in [12, Remark 6.3.6], it is proved that $\mathrm{Mod}(k_{X_{sa}})$ is equivalent to the category of sheaves on the site whose open sets are the open subanalytic subsets of X and whose coverings are the same as X_{sa}.

Let $\mathrm{PSh}(k_{X_{sa}})$ be the category of presheaves of k-modules on X_{sa}. Denote by $for : \mathrm{Mod}(k_{X_{sa}}) \to \mathrm{PSh}(k_{X_{sa}})$ the forgetful functor which associates to a sheaf F on X_{sa} its underlying presheaf. It is well known that for admits a left adjoint $\cdot^a : \mathrm{PSh}(k_{X_{sa}}) \to \mathrm{Mod}(k_{X_{sa}})$.

We denote by

$$\varrho : X \longrightarrow X_{sa} \ ,$$

the natural morphism of sites associated to $\mathrm{Op}^c(X_{sa}) \longrightarrow \mathrm{Op}(X)$. We refer to [12] for the definitions of the functors $\varrho_* : \mathrm{Mod}(k_X) \longrightarrow \mathrm{Mod}(k_{X_{sa}})$ and $\varrho^{-1} : \mathrm{Mod}(k_{X_{sa}}) \longrightarrow \mathrm{Mod}(k_X)$ and for Proposition 1.1.4 below.

Proposition 1.1.4. *(i) The functor ϱ^{-1} is left adjoint to ϱ_*.*

(ii) The functor ϱ^{-1} has a left adjoint denoted by $\varrho_! : \mathrm{Mod}(k_X) \to \mathrm{Mod}(k_{X_{sa}})$.

(iii) The functors ϱ^{-1} and $\varrho_!$ are exact, ϱ_* is exact on constructible sheaves.

(iv) The functors ϱ_* and $\varrho_!$ are fully faithful.

Through ϱ_*, we will consider $\mathrm{Mod}(k_X)$ as a subcategory of $\mathrm{Mod}(k_{X_{sa}})$.

The functor $\varrho_!$ is described as follows. If $U \in \mathrm{Op}^c(X_{sa})$ and $F \in \mathrm{Mod}(k_X)$, then $\varrho_!(F)$ is the sheaf on X_{sa} associated to the presheaf $U \mapsto F(\overline{U})$.

Now, we are going to recall the definition of $\mathbb{R}$-preconstructibility and $\mathbb{R}$-constructibility for sheaves on X_{sa}.

Denote by $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ the full triangulated subcategory of the bounded derived category of $\mathrm{Mod}(\mathbb{C}_X)$ consisting of complexes whose cohomology

groups are $\mathbb{R}$-constructible sheaves. In what follows, for $F \in D^b(\mathbb{C}_{X_{sa}})$ and $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, we set for short

$$R\mathcal{H}om_{\mathbb{C}_X}(G,F) := \varrho^{-1} R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(G,F) \in D^b(\mathbb{C}_X)$$

and

$$RHom_{\mathbb{C}_X}(G,F) := R\Gamma(X, R\mathcal{H}om_{\mathbb{C}_X}(G,F)) \ .$$

Definition 1.1.5. *Let $F \in D^b(\mathbb{C}_{X_{sa}})$.*

(i) We say that F is $\mathbb{R}$-preconstructible *if for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ with compact support and any $j \in \mathbb{Z}$,*

$$\dim_{\mathbb{C}} \mathrm{R}^j Hom_{\mathbb{C}_X}(G,F)) < +\infty \ .$$

(ii) We say that F is $\mathbb{R}$-constructible *if for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$,*

$$R\mathcal{H}om_{\mathbb{C}_X}(G,F) \in D^b_{\mathbb{R}-c}(\mathbb{C}_X) \ .$$

1.2 $\mathcal{D}$-modules and tempered holomorphic functions

For the general theory of $\mathcal{D}$-modules, we refer to [10] and [2]. For the basic results on the tempered De Rham functor we refer to [12]. For an introduction to derived categories, we refer to [11].

Let X be a complex analytic manifold. We denote by $\mathcal{O}_X$ the sheaf of rings of holomorphic functions on X and by $\mathcal{D}_X$ the sheaf of rings of linear partial differential operators with coefficients in $\mathcal{O}_X$.

Given two left $\mathcal{D}_X$-modules $\mathcal{M}_1, \mathcal{M}_2$, we denote by $\mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2$ the internal tensor product. Let us start by recalling the following

Proposition 1.2.1 ([10] Proposition 3.5)**.** *Let $\mathcal{N}$ be a right $\mathcal{D}_X$-module, $\mathcal{M}_1, \mathcal{M}_2$ left $\mathcal{D}_X$-modules. Then*

$$\mathcal{N} \underset{\mathcal{D}_X}{\otimes} (\mathcal{M}_1 \overset{D}{\otimes} \mathcal{M}_2) \simeq (\mathcal{N} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}_1) \underset{\mathcal{D}_X}{\otimes} \mathcal{M}_2 \ .$$

Now, let T^*X denote the cotangent bundle on X. We denote by $\mathrm{Mod}_c(\mathcal{D}_X)$ the full subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ whose objects are coherent over $\mathcal{D}_X$. For $\mathcal{M} \in \mathrm{Mod}_c(\mathcal{D}_X)$ we denote by $\mathrm{char}\mathcal{M}$ the characteristic variety of $\mathcal{M}$. Recall that $\mathrm{char}\mathcal{M} \subset T^*X$ and that $\mathcal{M}$ is said *holonomic* if $\mathrm{char}\mathcal{M}$ is Lagrangian. We denote by $\mathrm{Mod}_h(\mathcal{D}_X)$ the full subcategory of $\mathrm{Mod}(\mathcal{D}_X)$ consisting of holonomic modules.

We denote by $D^b_{coh}(\mathcal{D}_X)$ (resp. $D^b_h(\mathcal{D}_X)$) the full subcategory of $D^b(\mathcal{D}_X)$ consisting of bounded complexes whose cohomology groups are coherent (resp. holonomic) $\mathcal{D}_X$-modules. For $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$, set $\mathrm{char}\mathcal{M} := \cup_{j\in\mathbb{Z}} \mathrm{char} H^j(\mathcal{M})$.

Let $\pi_X : T^*X \to X$ be the canonical projection, $T^*_X X$ the zero section of T^*X and $\dot{T}^*X := T^*X \setminus T^*_X X$.

For $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$, set

$$S(\mathcal{M}) := \pi_X\left(\mathrm{char}\mathcal{M} \cap \dot{T}^*X\right) .$$

It is well known that, if $\mathcal{M} \in D^b_h(\mathcal{D}_X)$, then $S(\mathcal{M}) \neq X$ is a closed *analytic* subset of X. That is to say, for any $z \in S(\mathcal{M})$ there exist a neighbourhood W of z and finitely many functions $f_1, \ldots, f_k \in \mathcal{O}_X(W)$ such that $S(\mathcal{M}) \cap W = \{x \in W; f_1(x) = \ldots = f_k(x) = 0\}$.

Definition 1.2.2. *An object $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ is said* regular holonomic *if, for any $x \in X$,*

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_{X,x}) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, \widehat{\mathcal{O}}_{X,x}) ,$$

where $\widehat{\mathcal{O}}_{X,x}$ is the $\mathcal{D}_{X,x}$-module of formal power series at x. We denote by $D^b_{rh}(\mathcal{D}_X)$ the full subcategory of $D^b_h(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules.

Now, let Z be a closed analytic subset of X. Let $\mathcal{I}_Z$ be the coherent ideal consisting of the holomorphic functions vanishing on Z, we set

$$\begin{aligned} \Gamma_{[Z]}\mathcal{M} &:= \varinjlim_k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Z, \mathcal{M}) , \\ \Gamma_{[X\setminus Z]}\mathcal{M} &:= \varinjlim_k \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^k_Z, \mathcal{M}) . \end{aligned}$$

If $S \subset X$ can be written as $S = Z_1 \setminus Z_2$, for Z_1 and Z_2 closed analytic sets, then it can be proved that the following object is well defined

$$\Gamma_{[S]}\mathcal{M} := \Gamma_{[Z_1]}\Gamma_{[X\setminus Z_2]}\mathcal{M}$$

and that $\Gamma_{[S]}$ is a left exact functor. We denote by $R\Gamma_{[S]}$ the left derived functor of $\Gamma_{[S]}$.

Theorem 1.2.3 ([10], Theorem 3.29)**.** *(i) Let $S_1, S_2 \subset X$ be difference of closed analytic subsets of X. Then*

$$R\Gamma_{[S_1]}R\Gamma_{[S_2]}\mathcal{M} \simeq R\Gamma_{[S_1\cap S_2]}\mathcal{M} .$$

(ii) Let Z be a closed analytic subset of X. For any $\mathcal{M} \in D^b(\mathcal{D}_X)$ there exists a distinguished triangle

$$R\Gamma_{[Z]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow R\Gamma_{[X\setminus Z]}\mathcal{M} \xrightarrow{+1} .$$

Let us also recall the following fundamental

Theorem 1.2.4 ([10] Theorem 4.30)**.** *Let Z be a closed submanifold of X. Then the category of coherent $\mathcal{D}_X$-modules supported by Z is equivalent to the category of coherent $\mathcal{D}_Z$-modules.*

Given two complex analytic manifolds X and Y of dimension, respectively, d_X and d_Y and a holomorphic morphism $f : X \to Y$, we denote by $f_{\mathrm{D}}^{-1} : \mathrm{Mod}(\mathcal{D}_Y) \to \mathrm{Mod}(\mathcal{D}_X)$ the inverse image functor and by $\mathrm{D}f^{-1} : D^b(\mathcal{D}_Y) \to D^b(\mathcal{D}_X)$ its derived functor. Recall that f is said *smooth* if the corresponding maps of tangent spaces $T_xX \to T_{f(x)}Y$ are surjective for any $x \in X$. If f is a smooth map, then f_{D}^{-1} is an exact functor. For Proposition 1.2.5 below, we refer to Proposition 2.5.27 of [2] and to Proposition 3.35 of [10].

Proposition 1.2.5. *Let $Z \subset Y$ be an analytic set, $f : X \to Y$ a holomorphic map, $\mathcal{M} \in D^b_h(\mathcal{D}_Y)$. Then*

$$R\Gamma_{[f^{-1}(Z)]}(\mathcal{O}_X) \simeq \mathrm{D}f^{-1}(R\Gamma_{[Z]}\mathcal{O}_Y) \quad \textit{and} \quad R\Gamma_{[Z]}\mathcal{M} \simeq R\Gamma_{[Z]}\mathcal{O}_Y \overset{D}{\otimes} \mathcal{M} \ .$$

In particular,

$$R\Gamma_{[f^{-1}(Z)]}(\mathrm{D}f^{-1}\mathcal{M}) \simeq \mathrm{D}f^{-1}(R\Gamma_{[Z]}\mathcal{M}) \ .$$

Given $f \in \mathcal{O}_X$, let $Z := f^{-1}(0)$. We denote by $\mathcal{O}_X[*Z]$ the sheaf of meromorphic functions with poles on Z. Let us remark that $\mathcal{O}_X[*Z]$ is smooth over $\mathcal{O}_X$. Given $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$, we set $\mathcal{M}[*Z] := \mathcal{M} \overset{D}{\otimes} \mathcal{O}_X[*Z]$. One can prove that $R\Gamma_{[X\setminus Z]}\mathcal{M} \simeq \mathcal{M}[*Z]$.

We denote by $X_{\mathbb{R}}$ the real analytic manifold underlying X and by $\mathcal{D}_{X_{\mathbb{R}}}$ the sheaf of linear differential operators with real analytic coefficients. Furthermore, we denote by $\overline{X}$ the complex conjugate manifold, in particular $\mathcal{O}_{\overline{X}}$ is the sheaf of anti-holomorphic functions. Denote by $\mathcal{D}b_{X_{\mathbb{R}}}$ the sheaf of distributions on $X_{\mathbb{R}}$ and, for a closed subset Z of X, by $\Gamma_Z(\mathcal{D}b_{X_{\mathbb{R}}})$ the subsheaf of sections supported by Z. One denotes by $\mathcal{D}b^t_{X_{sa}}$ the presheaf of *tempered distributions* on $X_{\mathbb{R}}$ defined by

$$\mathrm{Op}^c(X_{sa}) \ni U \longmapsto \mathcal{D}b^t_{X_{sa}}(U) := \Gamma(X;\mathcal{D}b_{X_{\mathbb{R}}})\big/\Gamma_{X\setminus U}(X;\mathcal{D}b_{X_{\mathbb{R}}}) \ .$$

In [12] it is proved that $\mathcal{D}b^t_{X_{sa}}$ is a sheaf on X_{sa}. This sheaf is well defined in the category $\mathrm{Mod}(\varrho_!\mathcal{D}_X)$. Moreover, for any $U \in \mathrm{Op}^c(X_{sa})$, $\mathcal{D}b^t_{X_{sa}}$ is $\Gamma(U,\cdot)$-acyclic.

One defines the complex of sheaves $\mathcal{O}^t_{X_{sa}} \in D^b\big(\varrho_!\mathcal{D}_X\big)$ of tempered holomorphic functions as

$$\mathcal{O}^t_{X_{sa}} := R\mathcal{H}om_{\varrho_!\mathcal{D}_{\overline{X}}}\big(\varrho_!\mathcal{O}_{\overline{X}}, \mathcal{D}b^t_{X_{\mathbb{R}}}\big) \ .$$

Let Ω^j_X be the sheaf of differential forms of degree j and, for sake of simplicity, let us write Ω_X instead of $\Omega^{d_X}_X$.

Set

(1.1)
$$\Omega^t_X := \varrho_!\Omega_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{O}^t_{X_{sa}}\,, \qquad \Omega^{\mathcal{D}b^t}_X := \varrho_!\Omega_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t_{X_{sa}}\,, \qquad \Omega^{j,\mathcal{D}b^t}_X := \varrho_!\Omega^j_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t_{X_{sa}}\,.$$

Let us now define the De Rham functors over $\mathcal{D}_X$, let $\mathcal{M} \in D^b(\mathcal{D}_X)$,

$$\mathrm{DR}_{\mathcal{D}_X}\mathcal{M} := \Omega \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}\,, \quad DR^t_{\mathcal{D}_X}\mathcal{M} := \Omega^t_X \overset{L}{\underset{\varrho_!\mathcal{D}_X}{\otimes}} \varrho_!\mathcal{M}\,, \quad \mathrm{DR}^{\mathcal{D}b^t}_{\mathcal{D}_X}\mathcal{M} := \Omega^{\mathcal{D}b^t} \overset{L}{\underset{\varrho_!\mathcal{D}_X}{\otimes}} \varrho_!\mathcal{M}\,.$$

Furthermore, let us define the De Rham complexes over $\mathcal{O}_X$, let $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$,

$$\mathrm{DR}_{\mathcal{O}_X}\mathcal{M} := 0 \longrightarrow \mathcal{M} \xrightarrow{\nabla^{(0)}} \Omega^1_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \xrightarrow{\nabla^{(1)}} \Omega^2_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M} \longrightarrow \dots\,, \tag{1.2}$$

(1.3)
$$\mathrm{DR}^{\mathcal{D}b^t}_{\mathcal{O}_X}\mathcal{M} := 0 \longrightarrow \varrho_!\mathcal{M} \xrightarrow{\nabla^{(0)}} \Omega^{1,\mathcal{D}b^t}_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\mathcal{M} \xrightarrow{\nabla^{(1)}} \Omega^{2,\mathcal{D}b^t}_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\mathcal{M} \longrightarrow \dots\,,$$

where $\nabla^{(j)}$ is defined by the action of vector fields on $\mathcal{M}$.

The complex $\mathrm{DR}_{\mathcal{O}_X}$ is an object in the category of differential complexes. As we do not need the theory of differential complexes, we do not recall it here and we refer to [16] and to the references cited there. Moreover, the objects and the results recalled here on the De Rham functors can be stated in much more general settings, we refer to the bibliography.

Let $D^b_{\mathbb{C}-c}(\mathbb{C}_X)$ be the full subcategory of $D^b(\mathbb{C}_X)$ whose objects have $\mathbb{C}$-constructible cohomology groups ([11]).

Proposition 1.2.6 ([2], Proposition 2.2.10)**.** *Let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. The complexes $\mathrm{DR}_{\mathcal{O}_X}\mathcal{M}$ and $\mathrm{DR}_{\mathcal{D}_X}\mathcal{M}[-d_X]$ are isomorphic in the category $D^b_{\mathbb{C}-c}(\mathbb{C}_X)$.*

One can prove that the isomorphism of Proposition 1.2.6 extends to

$$\mathrm{DR}^{\mathcal{D}b^t}_{\mathcal{O}_X}\mathcal{M} \simeq \mathrm{DR}^{\mathcal{D}b^t}_{\mathcal{D}_X}\mathcal{M}[-d_X] \tag{1.4}$$

as objects in $D^b(\mathbb{C}_{X_{sa}})$.

Theorem 1.2.7 ([12] Theorem 7.4.12, Theorem 7.4.1)**.** *(i) Let $\mathcal{L} \in D^b_{rh}(\mathcal{D}_X)$ and set $L := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}, \mathcal{O}_X)$. There exists a natural isomorphism in $D^b(\mathbb{C}_{X_{sa}})$*

$$DR^t_{\mathcal{D}_X}\mathcal{L} \simeq R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(L, \Omega^t_X)\,.$$

(ii) Let $f : X \to Y$ be a holomorphic map and let $\mathcal{N} \in D^b(\mathcal{D}_Y)$. There is a natural isomorphism in $D^b(\mathbb{C}_{X_{sa}})$

$$DR^t_{\mathcal{D}_X}(\mathrm{D}f^{-1}\mathcal{N})[d_X] \xrightarrow{\sim} f^!(DR^t_{\mathcal{D}_Y}\mathcal{N})[d_Y] \ .$$

We are now going to recall a conjecture of M. Kashiwara and P. Schapira on constructibility of tempered holomorphic solutions of holonomic $\mathcal{D}$-modules and the results we obtained on curves.

For sake of shortness, we set

$$\mathscr{S}ol^t(\mathcal{M}) := R\mathcal{H}om_{\varrho_!\mathcal{D}_X}(\varrho_!\mathcal{M}, \mathcal{O}^t) \in D^b(\mathbb{C}_{X_{sa}}) \ .$$

Conjecture 1.2.8 ([13]). *Let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Then $\mathscr{S}ol^t(\mathcal{M}) \in D^b(\mathbb{C}_{X_{sa}})$ is $\mathbb{R}$-constructible.*

In [21], we proved that Conjecture 1.2.8 is true on analytic curves.

Theorem 1.2.9. *Let X be a complex curve and $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Then, $\mathscr{S}ol^t(\mathcal{M})$ is $\mathbb{R}$-constructible.*

Let $\mathcal{M} \in D^b_{coh}(\mathcal{D}_X)$. Set

$$\mathbb{D}_X := \mathcal{E}xt^{d_X}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}) \ .$$

Theorem 1.2.10. *(i) The functor $\mathbb{D}_X : \mathrm{Mod}_h(\mathcal{D}_X)^{op} \to \mathrm{Mod}_h(\mathcal{D}_X)$ is an equivalence of categories.*

(ii) Let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Then,

$$\mathscr{S}ol^t(\mathcal{M}) \simeq DR^t_{\mathcal{D}_X}(\mathbb{D}_X\mathcal{M}) \ .$$

1.3 Elementary asymptotic decomposition of meromorphic connections

In this subsection we are going to recall some fundamental results on the asymptotic decomposition of meromorphic connections. The first results on this subject were obtained by H. Majima ([17]) and C. Sabbah ([24]). In particular, Sabbah proved that any meromorphic connection admitting a *good* formal decomposition admits an asymptotic decompositions on small multisectors. Let us recall that Sabbah conjectured that any meromorphic connection admits a *good* formal decomposition after a finite number of pointwise blow-up and ramification maps ([25]). Such a conjecture was proved in the algebraic case by T. Mochizuki ([19, 20]) and in the analytic case by K. Kedlaya ([14, 15]). Let us also stress the fact that, in the works of Sabbah, Mochizuki and Kedlaya, there is an essential *goodness* property of formal

invariants which plays a central role in the study of the formal and asymptotic decompositions. As in this paper we are not concerned with the formal decomposition of meromorphic connections and since the goodness property is not needed within the scope of our results, we are not going to give details on them. In this way, we will avoid to go into technicalities unessential in the rest of the paper. For this subsection we refer to [25, 20, 4].

Let us start by recalling some results about integrable connections on an analytic manifold X of dimension n with meromorphic poles on a divisor Z. As in the rest of the paper we will just need the case where Z is a normal crossing hypersurface, from now on, we will suppose such hypothesis.

Let Ω^j_X be the sheaf of j-forms on X. Let $\mathcal{M}$ be a finitely generated $\mathcal{O}[*Z]$-module endowed with a $\mathbb{C}_X$-linear morphism $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ satisfying the Leibniz rule, that is to say, for any $h \in \mathcal{O}[*Z]$, $m \in \mathcal{M}$, $\nabla(hm) = dh \otimes m + h\nabla m$. The morphism ∇ induces $\mathbb{C}_X$-linear morphisms $\nabla^{(j)} : \Omega^j_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^{j+1}_X \otimes_{\mathcal{O}_X} \mathcal{M}$.

Definition 1.3.1. *A meromorphic flat connection on X with poles along Z is a locally free $\mathcal{O}[*Z]$-module of finite type $\mathcal{M}$ endowed with a $\mathbb{C}_X$-linear morphism ∇ as above such that $\nabla^{(1)} \circ \nabla = 0$.*

For sake of shortness, in the rest of the paper, we will drop the adjective "*flat*". If there is no risk of confusion, given a meromorphic connection $(\mathcal{M}, \nabla)$, we will simply denote it by $\mathcal{M}$.

Let $\mathcal{M}_1, \mathcal{M}_2$ be two coherent $\mathcal{O}_X[*Z]$-modules, a morphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ induces a morphism $\varphi' : \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}_1 \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}_2$. A *morphism of meromorphic connections* $(\mathcal{M}_1, \nabla_1) \to (\mathcal{M}_2, \nabla_2)$ is given by a morphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ of coherent $\mathcal{O}_X[*Z]$-modules such that $\varphi' \circ \nabla_1 = \nabla_2 \circ \varphi$. We denote by $\mathfrak{M}(X, Z)$ the *category of meromorphic connections with poles along Z*.

It is well known (see [2]) that, if Z is a normal crossing hypersurface, the image through the functor $\cdot \overset{D}{\otimes} \mathcal{O}[*Z]$ of the full subcategory of $\mathrm{Mod}_h(\mathcal{D}_X)$ of objects with singular support contained in Z, is equivalent to the category of meromorphic connections with poles along Z. In particular, if $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$ satisfies $S(\mathcal{M})$ is a normal crossing hypersurface and $\mathcal{M} \simeq R\Gamma_{[X \setminus S(\mathcal{M})]}\mathcal{M} \simeq \mathcal{M} \overset{D}{\otimes} \mathcal{O}[*S(\mathcal{M})]$, then the morphism $\nabla^{(0)} : \mathcal{M} \longrightarrow \Omega^1_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$, defined in (1.2), gives rise to a meromorphic connection. A meromorphic connection is said *regular* if it is regular as a $\mathcal{D}$-module. Furthermore the tensor product in $\mathfrak{M}(X, Z)$ is well defined and it coincides with the tensor product of $\mathcal{D}$-modules. With an abuse of language, given a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with singular support contained in Z, we will call $\mathcal{M}$ a meromorphic connection if $\mathcal{M} \simeq R\Gamma_{[X \setminus Z]}\mathcal{M} \simeq \mathcal{M} \overset{D}{\otimes} \mathcal{O}_X[*Z]$.

Let us recall that, if $(\mathcal{M}, \nabla) \in \mathfrak{M}(X, Z)$ and $\mathcal{M}$ is an $\mathcal{O}_X[*Z]$-module of rank r, then, in a given basis of local sections of $\mathcal{M}$, we can write ∇ as $d - A$

where A is a $r \times r$ matrix with entries in $\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{O}_X[*Z]$. Now, let X' be a complex manifold and $f : X' \to X$ a holomorphic map. Let us suppose that $Z' := f^{-1}(Z)$ has codimension 1 everywhere in X'. Then, we can define the inverse image $f^*\mathcal{M}$ of $\mathcal{M}$ on X' with poles along Z'. As an $\mathcal{O}_{X'}[*Z']$-module, it is $f^{-1}\mathcal{M}$ and the matrix of the connection in a local base is f^*A. With a harmless abuse of notation, we will write $f_{\mathrm{D}}^{-1}\mathcal{M}$ for $f^*\mathcal{M}$.

Let us now introduce the elementary asymptotic decompositions.

Let us denote by $\widetilde{X}$ the real oriented blow-up of the irreducible components of Z and by $\pi : \widetilde{X} \to X$ the composition of all these. Let us suppose that Z is locally defined by $x_1 \cdot \ldots \cdot x_k = 0$. Then, locally $\widetilde{X} \simeq (S^1 \times \mathbb{R}_{\geq 0})^k \times \mathbb{C}^{n-k}$. By a *multisector* we mean a set of the form $\prod_{j=1}^{k}(I_j \times V_j) \times W$ where $I_j \subset S^1$ is an open connected set, $V_j = [0, r_j[$ $(r_j > 0)$ and $W \subset \mathbb{C}^{n-k}$ is an open polydisc. Let $\overline{x}_1, \ldots, \overline{x}_n$ denote the antiholomorphic coordinates on X. Then $\partial_{\overline{x}_j}$ acts on $\mathcal{C}^\infty_{\widetilde{X}}$. The sheaf on $\tilde{X}(Z)$ of holomorphic functions with asymptotic development on Z, denoted $\mathcal{A}_{\widetilde{X}}$, is defined as

$$\mathcal{A}_{\widetilde{X}} := \bigcap_{j=1}^{k} \ker\Big(\overline{x}_j \partial_{\overline{x}_j} : \mathcal{C}^\infty_{\widetilde{X}(D)} \to \mathcal{C}^\infty_{\widetilde{X}(D)}\Big) \cap \bigcap_{j=k+1}^{n} \ker\Big(\partial_{\overline{x}_j} : \mathcal{C}^\infty_{\widetilde{X}(D)} \to \mathcal{C}^\infty_{\widetilde{X}(D)}\Big) \, .$$

The sections of $\mathcal{A}_{\widetilde{X}}$ are holomorphic functions on $\widetilde{X}$ which admit an asymptotic development in the sense of [17] (see also [25]).

Given $\mathcal{M} \in \mathfrak{M}(X, Z)$, we set $\mathcal{M}_{\widetilde{X}} := \mathcal{A}_{\widetilde{X}} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^{-1}\mathcal{M}$.

Definition 1.3.2. *(i) Let φ be a local section of $\mathcal{O}_X[*Z]/\mathcal{O}_X$, we denote by $\mathcal{L}^\varphi$ the meromorphic connection of rank 1 whose matrix in a basis is $d\varphi$.*

(ii) An elementary local model $\mathcal{M}$ *is a meromorphic connection isomorphic to a direct sum*

$$\underset{\alpha \in A}{\oplus} \mathcal{L}^{\varphi_\alpha} \otimes \mathcal{R}_\alpha \ ,$$

where A is a finite set, $(\varphi_\alpha)_{\alpha \in A}$ is a family of local sections of $\mathcal{O}_X[*Z]/\mathcal{O}_X$ and $(\mathcal{R}_\alpha)_{\alpha \in A}$ is a family of regular meromorphic connections.

(iii) We say that $(\mathcal{M}, \nabla) \in \mathfrak{M}(X, Z)$ admits an elementary $\mathcal{A}$-decomposition if for any $\vartheta \in \pi^{-1}(Z)$ there exist an elementary local model $(\mathcal{M}^{el}, \nabla^{el})$ and an isomorphism

$$(\mathcal{M}_{\widetilde{X},\vartheta}, \nabla) \simeq (\mathcal{M}^{el}_{\widetilde{X},\vartheta}, \nabla^{el}) \ .$$

In particular, if $\operatorname{rk}\mathcal{M} = r$*, for any* $\vartheta \in \pi^{-1}(Z)$*, there exists* $Y_\vartheta \in \mathrm{Gl}(r, \mathcal{A}_\vartheta)$ *such that the following diagram commutes*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_{\widetilde{X},\vartheta} & \xrightarrow{\nabla} & \mathcal{M}_{\widetilde{X},\vartheta} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \Omega^1_{\widetilde{X},\vartheta} & \xrightarrow{\nabla} & \mathcal{M}_{\widetilde{X},\vartheta} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \Omega^2_{\widetilde{X},\vartheta} & \xrightarrow{\nabla} & \cdots \\ & & \downarrow{\scriptstyle Y_\vartheta \cdot} & & \downarrow{\scriptstyle Y_\vartheta \cdot} & & \downarrow{\scriptstyle Y_\vartheta \cdot} & & \\ 0 & \longrightarrow & \mathcal{M}^{el}_{\widetilde{X},\vartheta} & \xrightarrow{\nabla^{el}} & \mathcal{M}^{el}_{\widetilde{X},\vartheta} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \Omega^1_{\widetilde{X},\vartheta} & \xrightarrow{\nabla^{el}} & \mathcal{M}^{el}_{\widetilde{X},\vartheta} \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \Omega^2_{\widetilde{X},\vartheta} & \xrightarrow{\nabla^{el}} & \cdots . \end{array}$$

Let us now choose local coordinates such that Z is defined by the equation $x_1 \cdot \ldots \cdot x_k = 0$. By a *ramification map fixing* Z we mean a map

$$\begin{array}{cccc} \varrho_l : & \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ & (t_1 \ldots, t_n) & \longmapsto & (t_1^l \ldots, t_k^l, t_{k+1} \ldots, t_n) \ , \end{array} \tag{1.5}$$

for some $l \in \mathbb{Z}_{>0}$.

The proof of Theorem 1.3.3 below is based on deep results of T. Mochizuki ([19, 20]), K. Kedlaya ([14, 15]) and C. Sabbah ([25, 24]), see also [5, 4] for a concise exposition.

Theorem 1.3.3. *Let* $(\mathcal{M}, \nabla) \in \mathfrak{M}(X, Z)$*. For any* $x_0 \in Z$ *there exist a neighbourhood* W *of* x_0 *and a finite sequence of pointwise blow-ups above* x_0*,* $\sigma : Y \to X$ *such that* $\sigma^{-1}(Z)$ *is a normal crossing divisor and there exists a ramification map* $\eta : X' \to Y$ *fixing* $\sigma^{-1}(Z)$ *such that* $(\eta \circ \sigma)^{-1}_{\mathrm{D}} \mathcal{M}|_W$ *admits an elementary* $\mathcal{A}$*-decomposition.*

2 $\mathbb{R}$-preconstructibility of the tempered De Rham complex for holonomic $\mathcal{D}$-modules

In this section we are going to prove the $\mathbb{R}$-preconstructibility of $DR^t_{\mathcal{D}_X}\mathcal{M} := \Omega^t_X \underset{\varrho_! \mathcal{D}_X}{\otimes} \varrho_! \mathcal{M}$ for X a complex analytic manifold and $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Such a result is a weaker version of Conjecture 1.2.8 on the $\mathbb{R}$-constructibility of $DR^t_{\mathcal{D}_X}\mathcal{M}$.

Recall that $F \in D^b(\mathbb{C}_{X_{sa}})$ is said $\mathbb{R}$-*preconstructible* if for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ with compact support and any $j \in \mathbb{Z}$,

$$\dim_{\mathbb{C}} \mathrm{R}^j \mathcal{H}om_{\mathbb{C}_X}(G, F) < +\infty \ .$$

The general statement will be given and proved in Subsection 2.3. Roughly speaking, the proof is based on an induction process on dim X. The first step of the induction follows from Theorem 1.2.9. In Subsections 2.1 and 2.2 we treat particular cases of the inductive step.

2.1 The case of modules supported on analytic sets

In this subsection we prove the following

Proposition 2.1.1. *Let X be a complex analytic manifold, $Z \neq X$ a closed analytic subset of X. Suppose that for any complex analytic manifold Y with* $\dim Y < \dim X$ *and for any $\mathcal{N} \in D^b_h(\mathcal{D}_Y)$, $DR^t_{\mathcal{D}_Y}\mathcal{N}$ is $\mathbb{R}$-preconstructible. Then, for any $\mathcal{M} \in D^b_h(\mathcal{D}_X)$, $DR^t_{\mathcal{D}_X}(R\Gamma_{[Z]}\mathcal{M})$ is $\mathbb{R}$-preconstructible.*

Before going into the proof of Proposition 2.1.1, let us recall few facts on complex analytic subsets of an analytic manifold X. Given an analytic set $Z \subset X$, a point $z \in Z$ is said *regular* if there exists a neighbourhood W of z such that $Z \cap W$ is a closed submanifold of W. It is well known that the set of regular points of a regular analytic set $Z \subset X$, denoted Z_{reg}, is a dense open subset of Z. Furthermore $Z \setminus Z_{reg}$ is a closed analytic set called the *singular part of Z.*

Proof. We want to prove that for any $j > 0$, $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, with compact support,

$$\dim_{\mathbb{C}} R^j\mathrm{Hom}_{\mathbb{C}_X}(G, DR^t_{\mathcal{D}_X}(R\Gamma_{[Z]}\mathcal{M})) < +\infty \ . \tag{2.1}$$

As $D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ is generated by objects of the form $\mathbb{C}_U$, for $U \in \mathrm{Op}^c(X_{sa})$, it is sufficient to prove (2.1) with $G \simeq \mathbb{C}_U$.

We have the following sequence of isomorphisms

$$\begin{aligned}
R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_! R\Gamma_{[Z]}\mathcal{M}) &\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} (\varrho_!\mathcal{M} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_! R\Gamma_{[Z]}\mathcal{O}_X)) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, (\Omega^t_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_! R\Gamma_{[Z]}\mathcal{O}_X) \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!\mathcal{M}) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(\mathbb{C}_Z, \Omega^t_X) \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!\mathcal{M}) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U \underset{\mathbb{C}_X}{\otimes} \mathbb{C}_Z, DR^t_{\mathcal{D}_X}\mathcal{M}) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_{U\cap Z}, DR^t_{\mathcal{D}_X}\mathcal{M}) \ .
\end{aligned}$$

We have used Proposition 1.2.1 in the second isomorphism and Theorem 1.2.7(i) in the third isomorphism.

In particular, it is sufficient to prove (2.1) with $G \simeq \mathbb{C}_{U\cap Z}$.

Let us now suppose that $Z \neq X$ is a closed submanifold of X. Set $d_Z :=$ $\dim Z$ and $d_X := \dim X$ Let us denote by $i_Z : Z \to X$ the inclusion. By Theorem 1.2.4, there exists $\mathcal{N} \in D^b_h(\mathcal{D}_Z)$ such that $\mathrm{D}i_Z{}^{-1}R\Gamma_{[Z]}\mathcal{M} \simeq \mathcal{N}$. Furthermore, for $V \in \mathrm{Op}^c(Z_{sa})$, one has $i_{Z!}\mathbb{C}_{Z,V} \simeq \mathbb{C}_{X,V}$. We can now write the following sequence of isomorphisms

$$\begin{aligned} R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_{X,V}, \Omega^t_X \otimes_{\mathcal{D}_X} \varrho_! R\Gamma_{[Z]}\mathcal{M}) &\simeq R\mathrm{Hom}_{\mathbb{C}_X}(i_{Z!}\mathbb{C}_{Z,V}, \Omega^t_X \otimes_{\mathcal{D}_X} \varrho_! R\Gamma_{[Z]}\mathcal{M}) \\ &\simeq R\mathrm{Hom}_{\mathbb{C}_Z}(\mathbb{C}_{Z,V}, i^!_Z(\Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_! R\Gamma_{[Z]}\mathcal{M})) \\ &\simeq R\mathrm{Hom}_{\mathbb{C}_Z}(\mathbb{C}_{Z,V}, DR^t_{\mathcal{D}_Z}(\mathrm{D}i_Z{}^{-1} R\Gamma_{[Z]}\mathcal{M})[d_X - d_Z]) \\ &\simeq R\mathrm{Hom}_{\mathbb{C}_Z}(\mathbb{C}_{Z,V}, DR^t_{\mathcal{D}_Z}\mathcal{N}[d_X - d_Z]) \ , \end{aligned}$$

where we have used Theorem 1.2.7 (ii) in the third isomorphism.

Hence by hypothesis we have that, if $Z \neq X$ is a closed submanifold of X, for any $j > 0$, $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, with compact support,

$$\dim_{\mathbb{C}} R^j\mathrm{Hom}_{\mathbb{C}_X}(G, DR^t_{\mathcal{D}_X}(R\Gamma_{[Z]}\mathcal{M})) < +\infty \ . \tag{2.2}$$

Let us treat now the case where $Z \neq X$ is a generic closed analytic subset of X.

Consider the following distinguished triangle

$$R\Gamma_{[Z\setminus Z_{reg}]}\mathcal{M} \longrightarrow R\Gamma_{[Z]}\mathcal{M} \longrightarrow R\Gamma_{[Z_{reg}]}\mathcal{M} \underset{+1}{\longrightarrow} .$$

It follows that it is sufficient to prove (2.1) for $Z = Z_{reg}$ and $Z = Z \setminus Z_{reg}$.

As the statement is local, the case of Z_{reg} is easily solved by reducing to the case of a closed submanifold treated above by choosing a convenient neighbourhood of Z_{reg} where Z_{reg} is closed.

The case of $Z \setminus Z_{reg}$ is treated by using induction on its dimension.

□

2.2 The case of meromorphic connections

In this subsection we are going to prove the following

Proposition 2.2.1. *Let X be a complex analytic manifold of dimension $n \geq 2$. Suppose that for any complex analytic manifold Y with $1 \leq \dim Y < n$ and for any $\mathcal{N} \in D^b_h(\mathcal{D}_Y)$, $DR^t_{\mathcal{D}_Y}\mathcal{N}$ is $\mathbb{R}$-preconstructible. Then for any $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ such that*

$$\mathcal{M} \simeq R\Gamma_{[X\setminus S(\mathcal{M})]}\mathcal{M} \ , \tag{2.3}$$

$DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible.

Before going into the proof of Proposition 2.2.1, let us make some standard reductions allowing us to add some hypothesis to the singular support of $\mathcal{M}$.

Lemma 2.2.2. *(i) Suppose that for any hypersurface $V \subset X$ and any $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ such that $\mathcal{M} \simeq R\Gamma_{[X\setminus V]}\mathcal{M}$, $DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible. Then for any $Z \neq X$ closed analytic subset of X and any $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ such that $\mathcal{M} \simeq R\Gamma_{[X\setminus Z]}\mathcal{M}$, $DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible.*

(ii) Suppose that for any complex analytic manifold Y, any $\mathcal{N} \in D^b_h(\mathcal{D}_Y)$ satisfying $\mathcal{N} \simeq R\Gamma_{[Y\setminus S(\mathcal{N})]}\mathcal{N}$ and such that $S(\mathcal{N})$ is a normal crossing hypersurface, $DR^t_{\mathcal{D}_Y}\mathcal{N}$ is $\mathbb{R}$-preconstructible. Then, for any $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ satisfying (2.3) and such that $S(\mathcal{M})$ is a hypersurface, $DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible.

Proof. (i). Locally $Z \neq X$ be a closed analytic subset of X. Then, locally, $Z = \{x \in X;\ f_1(x) = \ldots = f_l(x) = 0\}$, for some $f_j \in \mathcal{O}_X$, $j = 1, \ldots, l$. Set $Z_l := \{x \in X;\ f_l(x) = 0\}$ and $\hat{Z}_l := \{x \in X;\ f_1(x) = \ldots = f_{l-1}(x) = 0\}$. We have the following distinguished triangle (see [10, Lemma 3.19])

$$R\Gamma_{[X\setminus Z]}\mathcal{M} \longrightarrow R\Gamma_{[X\setminus Z_l]}\mathcal{M} \oplus R\Gamma_{[X\setminus \hat{Z}_l]}\mathcal{M} \longrightarrow R\Gamma_{[X\setminus \hat{Z}_l]}\mathcal{M} \xrightarrow{+1} .$$

By induction, we reduce to the case of Z a hypersurface which is assumed in the hypothesis.

(ii). Suppose that $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$. Let $Z := S(\mathcal{M})$ be a hypersurface. Then

$$R\Gamma_{[X\setminus Z]}\mathcal{M} \simeq \mathcal{O}_X[*Z] \overset{D}{\otimes} \mathcal{M} .$$

In particular $\mathcal{M}$ is a meromorphic connection. It follows that there exists a finite sequence of pointwise complex blow-up maps $\pi : X' \to X$ such that $\pi|_{X'\setminus\pi^{-1}(Z)}$ is a biholomorphism and $S(\mathrm{D}\pi^{-1}\mathcal{M})$ has normal crossings.

Now, given $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, we have the following sequence of isomorphisms

$$\begin{aligned}
R\mathrm{Hom}_{\mathbb{C}_X}(G, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} R\Gamma_{[X\setminus Z]}\mathcal{M}) &\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} (\varrho_!\mathcal{M} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_! R\Gamma_{[X\setminus Z]}\mathcal{O}_X))\\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G, (\Omega^t_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_! R\Gamma_{[X\setminus Z]}\mathcal{O}_X) \underset{\varrho_!\mathcal{D}_X}{\otimes} \mathcal{M})\\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G, R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(\mathbb{C}_{X\setminus Z}, \Omega^t_X) \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!\mathcal{M})\\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G \underset{\mathbb{C}_X}{\otimes} \mathbb{C}_{X\setminus Z}, DR^t_{\mathcal{D}_X}\mathcal{M})\\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(R\pi_!\, \pi^!(G \otimes \mathbb{C}_{X\setminus Z}), DR^t_{\mathcal{D}_X}\mathcal{M})\\
&\simeq R\mathrm{Hom}_{\mathbb{C}_{X'}}(\pi^!(G \otimes \mathbb{C}_{X\setminus Z}), \pi^!(DR^t_{\mathcal{D}_X}\mathcal{M}))\\
&\simeq R\mathrm{Hom}_{\mathbb{C}_{X'}}(\pi^!(G \otimes \mathbb{C}_{X\setminus Z}), DR^t_{\mathcal{D}_{X'}}(\mathrm{D}\pi^{-1}\mathcal{M})) .
\end{aligned}$$

We have used Proposition 1.2.1 in the second isomorphism and Theorem 1.2.7(i) in the third isomorphism.

The case of $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ can be reduced to the case $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$ treated above by standard techniques.

□

It follows that it is sufficient to prove Proposition 2.2.1 adding the hypothesis that $Z := S(\mathcal{M})$ is a normal crossing hypersurface. Then, Lemma 2.2.2 will imply the general statement of Proposition 2.2.1. In particular, if Z is an hypersurface, (2.3) reads as $\mathcal{M} \simeq \mathcal{M} \overset{D}{\otimes} \mathcal{O}[*Z]$ and we can treat $\mathcal{M}$ as a meromorphic connection.

Now, the rest of the proof of Proposition 2.2.1 is divided in several lemmas which can be roughly grouped in three steps. In the first part we prove the statement for elementary models. Essentially, the method consists in taking the inverse image of connections on curves and using Theorem 1.2.9. In the second part, we prove the statement for meromorphic connections with an elementary $\mathcal{A}$-decomposition using the first step and Theorem 1.3.3. In the last part we study the properties of the tempered De Rham functor under inverse image of complex blow-ups and ramification maps. In the end of this Subsection we collect all the lemmas proved and we conclude the proof of Proposition 2.2.1.

We start by treating the case of elementary models. Recall Definition 1.3.2 (ii). First we need the following

Lemma 2.2.3. *Let X, Y be two complex manifolds, $f : X \to Y$ a holomorphic map, $\mathcal{M} \in D^b_h(\mathcal{D}_Y)$. If $DR^t_{\mathcal{D}_Y}\mathcal{M}$ is $\mathbb{R}$-preconstructible, then $DR^t_{\mathcal{D}_X}(\mathrm{D}f^{-1}\mathcal{M})$ is.*

Proof. Set $d_X := \dim X$ and $d_Y := \dim Y$. Given $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, we have the following isomorphisms

$$\begin{aligned} R\mathrm{Hom}_{\mathbb{C}_X}(G, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!\mathrm{D}f^{-1}\mathcal{M}) &\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G, f^!(\Omega^t_Y \underset{\varrho_!\mathcal{D}_Y}{\otimes} \varrho_!\mathcal{M})[d_Y - d_X]) \\ &\simeq R\mathrm{Hom}_{\mathbb{C}_Y}(Rf_!G, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!\mathcal{M}[d_Y - d_X]) \ , \end{aligned}$$

where we have used Theorem 1.2.7(ii) in the first isomorphism.

The conclusion follows. □

Now, we can suppose that $X \simeq \mathbb{C}^n$, $n \geq 2$, $Z = \{(x_1, \dots, x_k) \in \mathbb{C}^n ; x_1 \cdot \ldots \cdot x_n = 0\}$. Recall that, for $\varphi \in \mathcal{O}_X[*Z]/\mathcal{O}_X$, we denote by $\mathcal{L}^\varphi$ the meromorphic connection of rank 1 whose matrix in a basis is $d\varphi$.

Lemma 2.2.4. *Suppose that for any complex analytic manifold Y with $1 \leq \dim\, Y < n$ and for any $\mathcal{N} \in D^b_h(\mathcal{D}_Y)$, $DR^t_{\mathcal{D}_Y}\mathcal{N}$ is $\mathbb{R}$-preconstructible. Let $\varphi \in \frac{\mathcal{O}_X[*Z]}{\mathcal{O}_X}$, then $DR^t_{\mathcal{D}_X}\mathcal{L}^\varphi$ is $\mathbb{R}$-preconstructible.*

Proof. If $n = 1$, the conclusion follows by Theorem 1.2.9.

First consider the map $f : \mathbb{C}^2 \setminus \{x_2 = 0\} \to \mathbb{C}$, $f(x_1, x_2) = \frac{x_1}{x_2}$. As $df \neq 0$, $\mathrm{D}f^{-1}\mathcal{L}^{1/x} \simeq f_{\mathrm{D}}^{-1}\mathcal{L}^{1/x} \simeq \mathcal{L}^{x_2/x_1}$ and it can be extended to $\mathbb{C}^2$. Using Theorem 1.2.9 and Lemma 2.2.3, we obtain that $DR^t_{\mathcal{D}_{\mathbb{C}^2}}\mathcal{L}^{x_2/x_1}$ is $\mathbb{R}$-preconstructible.

Given $\varphi \in \frac{\mathcal{O}_{\mathbb{C}^n}[*Z]}{\mathcal{O}_{\mathbb{C}^n}}$, one checks easily that there exist $p, q \in \mathbb{C}[x_1, \dots, x_n]$ satisfying the conditions

(i) $\varphi = \frac{p(x_1,\dots,x_n)}{q(x_1,\dots,x_n)}$,

(ii) if we define

$$g : \begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^2 \\ (y_1, \dots, y_n) & \longmapsto & (q(y_1, \dots, y_n), p(y_1, \dots, y_n)) \end{array} ,$$

then $S := \{x \in \mathbb{C}^n;\ \mathrm{rk}\, dg(x) < 2\} \neq X$.

Set $\mathcal{N} := \mathrm{D}g^{-1}\mathcal{L}^{x_2/x_1}$. Clearly $H^0\mathcal{N} \simeq g_{\mathrm{D}}^{-1}\mathcal{L}^{x_2/x_1} \simeq \mathcal{L}^{\varphi}$.

Now, by Lemma 2.2.3 and the $\mathbb{R}$-preconstructibility of $DR^t_{\mathcal{D}_{\mathbb{C}^2}}\mathcal{L}^{x_2/x_1}$ (resp. by Proposition 2.1.1) we have that $DR^t_{\mathcal{D}_{\mathbb{C}^n}}\mathcal{N}$ (resp. $DR^t_{\mathcal{D}_{\mathbb{C}^n}}R\Gamma_{[S]}\mathcal{N}$) is $\mathbb{R}$-preconstructible. Hence, by the following distinguished triangle

$$R\Gamma_{[S]}\mathcal{N} \longrightarrow \mathcal{N} \longrightarrow R\Gamma_{[X\setminus S]}\mathcal{N} \xrightarrow{+1} ,$$

we have that $DR^t_{\mathcal{D}_{\mathbb{C}^n}}R\Gamma_{[X\setminus S]}\mathcal{N}$ is $\mathbb{R}$-preconstructible. Remark that, as f is smooth outside S, $H^j\mathcal{N}$ has support in S, for $j \geq 1$. In particular, $R\Gamma_{[X\setminus S]}H^0\mathcal{N} \simeq R\Gamma_{[X\setminus S]}\mathcal{N}$. Hence $DR^t_{\mathcal{D}_{\mathbb{C}^n}}R\Gamma_{[X\setminus S]}H^0\mathcal{N}$ is $\mathbb{R}$-preconstructible too. Furthermore, by Proposition 2.1.1, $DR^t_{\mathcal{D}_{\mathbb{C}^n}}R\Gamma_{[S]}H^0\mathcal{N}$ is $\mathbb{R}$-preconstructible. Using the distinguished triangle

$$R\Gamma_{[S]}H^0\mathcal{N} \longrightarrow H^0\mathcal{N} \longrightarrow R\Gamma_{[X\setminus S]}H^0\mathcal{N} \xrightarrow{+1} ,$$

we have that $DR^t_{\mathcal{D}_{\mathbb{C}^n}}H^0\mathcal{N}$ is $\mathbb{R}$-preconstructible and the statement is proved. □

We conclude the first step of the proof Proposition 2.2.1 with the following

Lemma 2.2.5. *Let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ and $\mathcal{R} \in D^b_{rh}(\mathcal{D}_X)$. If $DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible, then $DR^t_{\mathcal{D}_X}(\mathcal{M} \overset{D}{\otimes} \mathcal{R})$ is. In particular, if for any complex analytic manifold Y with $1 \leq \dim Y < n$ and for any $\mathcal{N} \in D^b_h(\mathcal{D}_Y)$, $DR^t_{\mathcal{D}_Y}\mathcal{N}$ is $\mathbb{R}$-preconstructible, then for any elementary model $\mathcal{M}$, $DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible.*

Proof. Let $L := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{R}, \mathcal{O}_X) \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$. For $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$, consider the following sequence of isomorphisms

$$\begin{aligned}
R\mathrm{Hom}_{\mathbb{C}_X}(G, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!(\mathcal{M} \overset{D}{\otimes} \mathcal{R})) &\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G, (\Omega^t_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\mathcal{R}) \underset{\varrho_!\mathcal{D}_X}{\otimes} \mathcal{M}) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G, R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(L, \Omega^t_X) \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!\mathcal{M}) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G, R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(L, DR^t_{\mathcal{D}_X}\mathcal{M})) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(G \underset{\mathbb{C}_X}{\otimes} L, DR^t_{\mathcal{D}_X}\mathcal{M}) \ .
\end{aligned}$$

In the previous series of isomorphisms we have used Proposition 1.2.1 in the first isomorphism and Theorem 1.2.7(i) in the second isomorphism. Hence, the first part of the statement is proved.

To conclude the proof, it is sufficient to combine the first part of the statement with Lemma 2.2.4. □

Now, we go into the second step of the proof of Proposition 2.2.1. We consider meromorphic connections with an elementary $\mathcal{A}$-decomposition.

Lemma 2.2.6. *Suppose that for any complex analytic manifold Y with $1 \leq \dim Y < n$ and for any $\mathcal{N} \in D^b_h(\mathcal{D}_Y)$, $DR^t_{\mathcal{D}_Y}\mathcal{N}$ is $\mathbb{R}$-preconstructible. Let $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$ be such that*

(i) $S(\mathcal{M})$ is a normal crossing hypersurface,

(ii) $\mathcal{M} \simeq R\Gamma_{[X \setminus S(\mathcal{M})]}\mathcal{M}$,

(iii) $\mathcal{M}$ admits an elementary $\mathcal{A}$-decomposition as a meromorphic connection.

Then $DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible.

Proof. We want to prove that for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ with compact support, $j \in \mathbb{Z}$

$$\dim R^j\mathrm{Hom}_{\mathbb{C}_{X_{sa}}}(G, DR^t_{\mathcal{D}_X}\mathcal{M}) < +\infty \ . \tag{2.4}$$

First, let us remark that it is sufficient to prove (2.4) for $G = \mathbb{C}_U$ for $U \in \mathrm{Op}^c(X_{sa})$. Then, we have the following sequence of isomorphisms

$$\begin{aligned}
R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} R\Gamma_{[X\setminus Z]}\mathcal{M}) &\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, \Omega^t_X \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!(\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X[*Z])) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, (\Omega^t_X \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\mathcal{O}_X[*Z]) \underset{\varrho_!\mathcal{D}_X}{\otimes} \mathcal{M}) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U, R\mathcal{H}om_{\mathbb{C}_{X_{sa}}}(\mathbb{C}_{X\setminus Z}, \Omega^t_X) \underset{\varrho_!\mathcal{D}_X}{\otimes} \varrho_!\mathcal{M}) \\
&\simeq R\mathrm{Hom}_{\mathbb{C}_X}(\mathbb{C}_U \underset{\mathbb{C}_X}{\otimes} \mathbb{C}_{X\setminus Z}, DR^t_{\mathcal{D}_X}\mathcal{M}) \ .
\end{aligned}$$

Hence, it is sufficient to prove (2.4) for $G = \mathbb{C}_V$, $V \in \mathrm{Op}^c(X_{sa})$, $V \subset X \setminus Z$.

In particular, for any $V \subset X \setminus Z$, there exists a finite family of open multisectors $\{S_j\}_{j\in J}$ such that $V \subset \bigcup_{j\in J} S_j$.

Now, let $\Omega_X^\bullet$ be the complex of differential forms on X. Recall definition (1.3) and the isomorphism (1.4). Since, for $U \in \mathrm{Op}^c(X_{sa})$, $\mathcal{D}b^t$ is $\Gamma(U,\cdot)$-acyclic, we have that $\Gamma(U, DR_{\mathcal{D}_X}^{\mathcal{D}b^t}\mathcal{M})$ is quasi-isomorphic to the complex

$$0 \longrightarrow \varrho_!\mathcal{M}(U) \xrightarrow{\nabla^{(0)}} \varrho_!\mathcal{M} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\Omega_X^1 \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t(U) \xrightarrow{\nabla^{(1)}} \varrho_!\mathcal{M} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\Omega_X^2 \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t(U) \xrightarrow{\nabla^{(2)}} \cdots .$$

Suppose that $\mathrm{rk}\mathcal{M} = r$. As $\mathcal{M}$ has an elementary $\mathcal{A}$-decomposition, for any small enough multisector S there exists $Y_S \in \mathrm{G}l(r, \mathcal{A}(S))$ and an elementary model $(\mathcal{M}^{el}, \nabla^{el})$ giving a quasi-isomorphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}(S) \arrow[r, "\nabla^{(0)}"] \arrow[d, "Y_S"] & \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega_X^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{A}(S) \arrow[r, "\nabla^{(1)}"] \arrow[d, "Y_S"] & \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega_X^2 \underset{\mathcal{O}_X}{\otimes} \mathcal{A}(S) \arrow[r, "\nabla^{(2)}"] \arrow[d, "Y_S"] & \cdots \\
0 \arrow[r] & \mathcal{M}^{el}(S) \arrow[r, "\nabla^{el,(0)}"] & \mathcal{M}^{el} \underset{\mathcal{O}_X}{\otimes} \Omega_X^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{A}(S) \arrow[r, "\nabla^{el,(1)}"] & \mathcal{M}^{el} \underset{\mathcal{O}_X}{\otimes} \Omega_X^2 \underset{\mathcal{O}_X}{\otimes} \mathcal{A}(S) \arrow[r, "\nabla^{el,(2)}"] & \cdots .
\end{tikzcd}$$

For sake of shortness, in the diagram above, we have omitted the notation of Definition 1.3.2 relative to the real blow-up $\widetilde{X}$.

Now, let $U \in \mathrm{Op}^c(X_{sa})$ be contained in a sufficiently small multisector. As the restriction of Y_S to U respects $\mathcal{D}b^t(U)$, we have the following quasi-isomorphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & \varrho_!\mathcal{M}(U) \arrow[r, "\nabla^{(0)}"] \arrow[d, "Y_S|_U"] & \varrho_!\mathcal{M} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\Omega_X^1 \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t(U) \arrow[r, "\nabla^{(1)}"] \arrow[d, "Y_S|_U"] & \varrho_!\mathcal{M} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\Omega_X^2 \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t(U) \arrow[r, "\nabla^{(2)}"] \arrow[d, "Y_S|_U"] & \cdots \\
0 \arrow[r] & \varrho_!\mathcal{M}^{el}(U) \arrow[r, "\nabla^{el,(0)}"] & \varrho_!\mathcal{M}^{el} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\Omega_X^1 \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t(U) \arrow[r, "\nabla^{el,(1)}"] & \varrho_!\mathcal{M}^{el} \underset{\varrho_!\mathcal{O}_X}{\otimes} \varrho_!\Omega_X^2 \underset{\varrho_!\mathcal{O}_X}{\otimes} \mathcal{D}b^t(U) \arrow[r, "\nabla^{el,(2)}"] & \cdots .
\end{tikzcd}$$

The second row of the above diagram is isomorphic to $\mathcal{D}R_{\mathcal{O}_X}^{\mathcal{D}b^t}(\mathcal{M}^{el})$.

By the isomorphism (1.4), we have

$$DR_{\mathcal{D}_X}^{\mathcal{D}b^t}\mathcal{M}(U) \simeq DR_{\mathcal{D}_X}^{\mathcal{D}b^t}\mathcal{M}^{el}(U) . \tag{2.5}$$

Sheafifying (2.5) we have that

$$\left(DR_{\mathcal{D}_X}^{\mathcal{D}b^t}\mathcal{M}\right)_S \simeq \left(DR_{\mathcal{D}_X}^{\mathcal{D}b^t}\mathcal{M}^{el}\right)_S . \tag{2.6}$$

Remark that the isomorphisms (2.6) depend on S and they can't be glued in a global isomorphism.

Now, taking the solutions of the Cauchy-Riemann system (i.e. applying the functor $\cdot \underset{\varrho_! \mathcal{D}_{\overline{X}}}{\otimes} \varrho_! \mathcal{O}_{\overline{X}}$ to (2.6)) we have

$$\left(DR^t_{\mathcal{D}_X} \mathcal{M}\right)_S \simeq \left(DR^t_{\mathcal{D}_X} \mathcal{M}^{el}\right)_S . \tag{2.7}$$

The $\mathbb{R}$-preconstructibility of the right hand side of (2.7) following from Lemma 2.2.5, the proof is complete.

□

Let us now study the behaviour of $\mathbb{R}$-preconstructibility of the tempered De Rham complex under inverse image in the case of a composition of complex blow-ups and ramification maps.

Lemma 2.2.7. *Let $Z \neq X$ be a closed analytic subset of X and $\pi : X' \to X$ be a composition of pointwise complex blow-ups above Z and ramification maps fixing Z. Let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ be such that $\mathcal{M} \simeq R\Gamma_{[X \setminus Z]}\mathcal{M}$. If $DR^t_{\mathcal{D}_{X'}}(\mathrm{D}\pi^{-1}\mathcal{M})$ is $\mathbb{R}$-preconstructible, then $DR^t_{\mathcal{D}_X}\mathcal{M}$ is.*

Proof. As concern a complex blow-up, the proof goes exactly as that of Lemma 2.2.2 *(ii)*. As concern a ramification map ϱ_l as given in (1.5), the proof goes as in Lemma 2.2.2 *(ii)* taking care of decomposing the sheaf G on the elements of a finite covering on which ϱ_l is an isomorphism.

□

Proof of Proposition 2.2.1

Let $\mathcal{M} \in D^b_h(\mathcal{D}_X)$ be such that $\mathcal{M} \simeq R\Gamma_{[X \setminus S(\mathcal{M})]}\mathcal{M}$. By Lemma 2.2.2 we can suppose that $S(\mathcal{M})$ is a normal crossing hypersurface. Furthermore, by standard techniques, we can assume that $\mathcal{M} \in \mathrm{Mod}_h(\mathcal{D}_X)$. In particular $\mathcal{M}$ can be considered as a meromorphic connection.

By Theorem 1.3.3, there exists a finite sequence of complex blow-ups and ramification maps π such that, $\pi_{\mathrm{D}}^{-1}\mathcal{M}$ admits an elementary $\mathcal{A}$-decomposition. It follows, by Lemma 2.2.7, that we can suppose that $\mathcal{M}$ admits an elementary $\mathcal{A}$-decomposition.

Then, we can conclude by Lemma 2.2.6.

□

2.3 The general statement

Let us recall that, given $\mathcal{M} \in D^b_h(\mathcal{D}_X)$,

$$DR^t_{\mathcal{D}_X}\mathcal{M} := \Omega^t_X \underset{\varrho_! \mathcal{D}_X}{\overset{L}{\otimes}} \varrho_! \mathcal{M} .$$

Moreover, $F \in D^b(\mathbb{C}_{X_{sa}})$ is said $\mathbb{R}$-preconstructible if for any $G \in D^b_{\mathbb{R}-c}(\mathbb{C}_X)$ with compact support and any $j \in \mathbb{Z}$,

$$\dim_{\mathbb{C}} \mathrm{R}^j Hom_{\mathbb{C}_X}(G, F) < +\infty .$$

We can now state and prove

Theorem 2.3.1. *Let X be a complex analytic manifold, $\mathcal{M} \in D^b_h(\mathcal{D}_X)$. Then $DR^t_{\mathcal{D}_X}\mathcal{M} \in D^b_X(\mathbb{C}_{X_{sa}})$ is $\mathbb{R}$-preconstructible.*

Proof. Let us use induction on the dimension of X.

If X is a curve, then the result follows at once from Theorem 1.2.9.

Now suppose that dim $X > 1$ and that the inductive hypothesis holds. That is, for any complex analytic manifold Y with $1 \leq \dim Y < \dim X$ and for any $\mathcal{N} \in D^b_h(\mathcal{D}_Y)$, $DR^t_{\mathcal{D}_Y}\mathcal{N}$ is $\mathbb{R}$-preconstructible. Let us prove that for any $\mathcal{M} \in D^b_h(\mathcal{D}_X)$, $DR^t_{\mathcal{D}_X}\mathcal{M}$ is $\mathbb{R}$-preconstructible.

By using the distinguished triangle

$$R\Gamma_{[S(\mathcal{M})]}\mathcal{M} \longrightarrow \mathcal{M} \longrightarrow R\Gamma_{[X\setminus S(\mathcal{M})]}\mathcal{M} \xrightarrow{+1},$$

it is enough to prove the statement for $R\Gamma_{[S(\mathcal{M})]}\mathcal{M}$ and for $R\Gamma_{[X\setminus S(\mathcal{M})]}\mathcal{M}$. The conclusion follows by Propositions 2.1.1 and 2.2.1

□

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